41.(a) Recall the derivative formula

$$\frac{d^n}{dx^n}(uv) = \binom{n}{0} v \frac{d^n u}{dx^n} + \binom{n}{1} \frac{dv}{dx} \frac{d^{n-1}u}{dx^{n-1}} + \ldots + \binom{n}{n} \frac{d^n v}{dx^n} u \, .$$

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4.3

2. The general solution of the homogeneous equation is $y_c = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$. Let $g_1(t) = 3t$ and $g_2(t) = \cos t$. By inspection, we find that $Y_1(t) = -3t$. Since $g_2(t)$ is a solution of the homogeneous equation, set $Y_2(t) = t(A \cos t + B \sin t)$. Substitution into the given ODE and comparing the coefficients of similar term results in $A = 0$ and $B = -1/4$. Hence the general solution of the nonhomogeneous problem is $y(t) = y_c(t) - 3t - t \sin t/4$.

3. The characteristic equation corresponding to the homogeneous problem can be written as $(r+1)(r^2+1)=0$. The solution of the homogeneous equation is $y_c = c_1e^{-t} + c_2\cos t + c_3\sin t$. Let $g_1(t) = e^{-t}$ and $g_2(t) = 4t$. Since $g_1(t)$ is a solution of the homogeneous equation, set $Y_1(t) = Ate^{-t}$. Substitution into the ODE results in $A = 1/2$. Now let $Y_2(t) = Bt + C$. We find that $B = -C = 4$. Hence the general solution of the nonhomogeneous problem is $y(t) = y_c(t) + te^{-t}/2 + 4(t-1)$.

4. The characteristic equation corresponding to the homogeneous problem can be written as $r(r+1)(r-1) = 0$. The solution of the homogeneous equation is $y_c = c_1 + c_2e^t + c_3e^{-t}$. Since $g(t) = 2\sin t$ is not a solution of the homogeneous problem, we can set $Y(t) = A\cos t + B\sin t$. Substitution into the ODE results in $A = 1$ and $B = 0$. Thus the general solution is $y(t) = c_1 + c_2e^t + c_3e^{-t} + \cos t$.

6. The characteristic equation corresponding to the homogeneous problem can be written as $(r^2+1)^2 = 0$. It follows that $y_c = c_1\cos t + c_2\sin t + t(c_3\cos t + c_4\sin t)$. Since $g(t)$ is not a solution of the homogeneous problem, set $Y(t) = A + B\cos 2t + C\sin 2t$. Substitution into the ODE results in $A = 3$, $B = 1/9$, $C = 0$. Thus the general solution is $y(t) = y_c(t) + 3 + \cos 2t/9$.

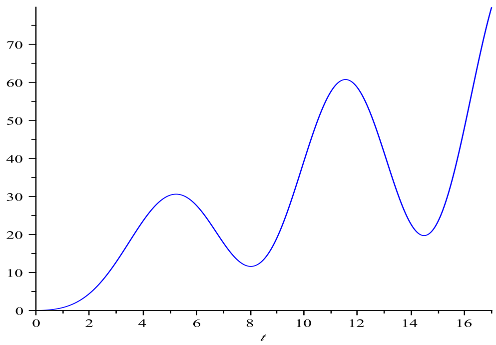
7. The characteristic equation corresponding to the homogeneous problem can be written as $r^3(r^3+1) = 0$. Thus the homogeneous solution is

$$y_c = c_1 + c_2\,t + c_3t^2 + c_4e^{-t} + e^{t/2}\left[c_5\cos(\sqrt{3}\,t/2) + c_5\sin(\sqrt{3}\,t/2)\right].$$

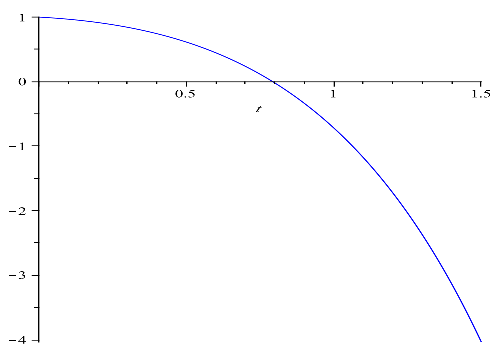
Note the $g(t) = t$ is a solution of the homogenous problem. Consider a particular solution of the form $Y(t) = t^3(At+B)$. Substitution into the ODE gives us that $A = 1/24$ and $B = 0$. Thus the general solution is $y(t) = y_c(t) + t^4/24$.

8. The characteristic equation corresponding to the homogeneous problem can be written as $r^3(r+1) = 0$. Hence the homogeneous solution is $y_c = c_1 + c_2\,t + c_3t^2 + c_4e^{-t}$. Since $g(t)$ is not a solution of the homogeneous problem, set $Y(t) = A\cos 2t + B\sin 2t$. Substitution into the ODE results in $A = 1/40$ and $B = 1/20$. Thus the general solution is $y(t) = y_c(t) + (\cos 2t + 2\sin 2t)/40$.

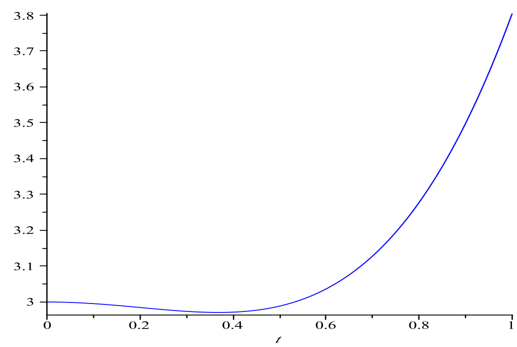
10. From Problem 22 in Section 4.2, the homogeneous solution is $y_c = c_1\cos t + c_2\sin t + t\,[c_3\cos t + c_4\sin t]$. Since $g(t)$ is not a solution of the homogeneous problem, substitute $Y(t) = At + B$ into the ODE to obtain $A = 3$ and $B = 4$. Thus the general solution is $y(t) = y_c(t) + 3t + 4$. Invoking the initial conditions, we find that $c_1 = -4$, $c_2 = -4$, $c_3 = 1$, $c_4 = -3/2$. Therefore the solution of the initial value problem is $y(t) = (t-4)\cos t - (3t/2+4)\sin t + 3t + 4$.



11. The characteristic equation can be written as $r(r^2 - 3r + 2) = 0$. Hence the homogeneous solution is $y_c = c_1 + c_2 e^t + c_3 e^{2t}$. Let $g_1(t) = e^t$ and $g_2(t) = t$. Note that g_1 is a solution of the homogeneous problem. Set $Y_1(t) = Ate^t$. Substitution into the ODE results in $A = -1$. Now let $Y_2(t) = Bt^2 + Ct$. Substitution into the ODE results in $B = 1/4$ and $C = 3/4$. Therefore the general solution is $y(t) = c_1 + c_2 e^t + c_3 e^{2t} - te^t + (t^2 + 3t)/4$. Invoking the initial conditions, we find that $c_1 = 1$, $c_2 = c_3 = 0$. The solution of the initial value problem is $y(t) = 1 - te^t + (t^2 + 3t)/4$.



12. The characteristic equation can be written as $(r - 1)(r + 3)(r^2 + 4) = 0$. Hence the homogeneous solution is $y_c = c_1 e^t + c_2 e^{-3t} + c_3 \cos 2t + c_4 \sin 2t$. None of the terms in $g(t)$ is a solution of the homogeneous problem. Therefore we can assume a form $Y(t) = Ae^{-t} + B \cos t + C \sin t$. Substitution into the ODE results in the values $A = 1/20$, $B = -2/5$, $C = -4/5$. Hence the general solution is $y(t) = c_1 e^t + c_2 e^{-3t} + c_3 \cos 2t + c_4 \sin 2t + e^{-t}/20 - (2 \cos t + 4 \sin t)/5$. Invoking the initial conditions, we find that $c_1 = 81/40$, $c_2 = 73/520$, $c_3 = 77/65$, $c_4 = -49/130$.



14. From Problem 4, the homogeneous solution is $y_c = c_1 + c_2 e^t + c_3 e^{-t}$. Consider the terms $g_1(t) = te^{-t}$ and $g_2(t) = 2\cos t$. Note that since $r = -1$ is a simple root of the characteristic equation, we set $Y_1(t) = t(At + B)e^{-t}$. The function $2\cos t$ is not a solution of the homogeneous equation. We set $Y_2(t) = C\cos t + D\sin t$. Hence the particular solution has the form $Y(t) = t(At + B)e^{-t} + C\cos t + D\sin t$.

15. The characteristic equation can be written as $(r^2 - 1)^2 = 0$. The roots are given as $r = \pm 1$, each with multiplicity two. Hence the solution of the homogeneous problem is $y_c = c_1 e^t + c_2 te^t + c_3 e^{-t} + c_4 te^{-t}$. Let $g_1(t) = e^t$ and $g_2(t) = \sin t$. The function e^t is a solution of the homogeneous problem. Since $r = 1$ has multiplicity *two*, we set $Y_1(t) = At^2 e^t$. The function $\sin t$ is not a solution of the homogeneous equation. We can set $Y_2(t) = B\cos t + C\sin t$. Hence the particular solution has the form $Y(t) = At^2 e^t + B\cos t + C\sin t$.

16. The characteristic equation can be written as $r^2(r^2 + 4) = 0$, and the roots are $r = 0, \pm 2i$. The root $r = 0$ has multiplicity *two*, hence the homogeneous solution is $y_c = c_1 + c_2 t + c_3\cos 2t + c_4 \sin 2t$. The functions $g_1(t) = \sin 2t$ and $g_2(t) = 4$ are solutions of the homogenous equation. The complex roots have multiplicity one, therefore we need to set $Y_1(t) = At\cos 2t + Bt\sin 2t$. Now $g_2(t) = 4$ is associated with the double root $r = 0$, so we set $Y_2(t) = Ct^2$. Finally, $g_3(t) = te^t$ (and its derivatives) is independent of the homogeneous solution. Therefore set $Y_3(t) = (Dt + E)e^t$. Conclude that the particular solution has the form $Y(t) = At\cos 2t + Bt\sin 2t + Ct^2 + (Dt + E)e^t$.

18. The characteristic equation can be written as $r^2(r^2 + 2r + 2) = 0$, with roots $r = 0$, with multiplicity two, and $r = -1 \pm i$. This means that the homogeneous solution is $y_c = c_1 + c_2 t + c_3 e^{-t}\cos t + c_4 e^{-t}\sin t$. The function $g_1(t) = 3e^t + 2te^{-t}$, and all of its derivatives, is independent of the homogeneous solution. Therefore set $Y_1(t) = Ae^t + (Bt + C)e^{-t}$. Now $g_2(t) = e^{-t}\sin t$ is a solution of the homogeneous equation, associated with the complex roots. We need to set $Y_2(t) = t(D\,e^{-t}\cos t + E\,e^{-t}\sin t)$. It follows that the particular solution has the form $Y(t) = Ae^t + (Bt + C)e^{-t} + t(D\,e^{-t}\cos t + E\,e^{-t}\sin t)$.

19. Differentiating $y = u(t)v(t)$, successively, we have

$$\begin{aligned} y' &= u'v + uv' \\ y'' &= u''v + 2u'v' + uv'' \\ &\vdots \\ y^{(n)} &= \sum_{j=0}^{n} \binom{n}{j} u^{(n-j)} v^{(j)} \end{aligned}$$

Setting $v(t) = e^{\alpha t}$, $v^{(j)} = \alpha^j e^{\alpha t}$. So for any $p = 1, 2, \ldots, n$,

$$y^{(p)} = e^{\alpha t} \sum_{j=0}^{p} \binom{p}{j} \alpha^j \, u^{(p-j)}.$$

It follows that

$$L\left[e^{\alpha t} u\right] = e^{\alpha t} \sum_{p=0}^{n} \left[a_{n-p} \sum_{j=0}^{p} \binom{p}{j} \alpha^j \, u^{(p-j)} \right] \qquad (*).$$

It is evident that the right hand side of Eq. $(*)$ is of the form

$$e^{\alpha t} \left[k_0 \, u^{(n)} + k_1 \, u^{(n-1)} + \ldots + k_{n-1} u' + k_n \, u \right].$$

Hence the operator equation $L\left[e^{\alpha t} u\right] = e^{\alpha t}(b_0 \, t^m + b_1 \, t^{m-1} + \ldots + b_{m-1} t + b_m)$ can be written as

$$k_0 \, u^{(n)} + k_1 \, u^{(n-1)} + \ldots + k_{n-1} u' + k_n \, u = b_0 \, t^m + b_1 \, t^{m-1} + \ldots + b_{m-1} t + b_m \, .$$

The coefficients k_i, $i = 0, 1, \ldots, n$ can be determined by collecting the like terms in the double summation in Eq. $(*)$. For example, k_0 is the coefficient of $u^{(n)}$. The only term that contains $u^{(n)}$ is when $p = n$ and $j = 0$. Hence $k_0 = a_0$. On the other hand, k_n is the coefficient of $u(t)$. The inner summation in $(*)$ contains terms with u, given by $\alpha^p u$ (when $j = p$), for each $p = 0, 1, \ldots, n$. Hence

$$k_n = \sum_{p=0}^{n} a_{n-p} \, \alpha^p \, .$$

21.(a) Clearly, e^{2t} is a solution of $y' - 2y = 0$, and te^{-t} is a solution of the differential equation $y'' + 2y' + y = 0$. The latter ODE has characteristic equation $(r+1)^2 = 0$. Hence $(D-2)\left[3e^{2t}\right] = 3(D-2)\left[e^{2t}\right] = 0$ and $(D+1)^2\left[te^{-t}\right] = 0$. Furthermore, we have $(D-2)(D+1)^2\left[te^{-t}\right] = (D-2)\left[0\right] = 0$, and $(D-2)(D+1)^2\left[3e^{2t}\right] = (D+1)^2(D-2)\left[3e^{2t}\right] = (D+1)^2\left[0\right] = 0$.

(b) Based on part (a),

$$(D-2)(D+1)^2\left[(D-2)^3(D+1)Y\right] = (D-2)(D+1)^2\left[3e^{2t} - te^{-t}\right] = 0,$$

since the operators are linear. The implied operations are associative and commutative. Hence $(D-2)^4(D+1)^3 Y = 0$. The operator equation corresponds to the solution of a linear homogeneous ODE with characteristic equation $(r-2)^4(r+1)^3 = 0$. The roots are $r = 2$, with multiplicity 4 and $r = -1$, with multiplicity 3. It follows that the given homogeneous solution is $Y(t) = c_1 e^{2t} + c_2 te^{2t} + c_3 t^2 e^{2t} + c_4 t^3 e^{2t} + c_5 e^{-t} + c_6 te^{-t} + c_7 t^2 e^{-t}$, which is a linear combination of seven independent solutions.

22. (15) Observe that $(D-1)\,[e^t] = 0$ and $(D^2+1)\,[\sin\, t] = 0$. Hence the operator $H(D) = (D-1)(D^2+1)$ is an annihilator of $e^t + \sin\, t$. The operator corresponding to the left hand side of the given ODE is $(D^2-1)^2$. It follows that

$$(D+1)^2(D-1)^3(D^2+1)Y = 0\,.$$

The resulting ODE is homogeneous, with solution $Y(t) = c_1 e^{-t} + c_2 te^{-t} + c_3 e^t + c_4 te^t + c_5 t^2 e^t + c_6 \cos\, t + c_7 \sin\, t$. After examining the homogeneous solution of Problem 15, and eliminating duplicate terms, we have $Y(t) = c_5 t^2 e^t + c_6 \cos\, t + c_7 \sin\, t$.

22. (16) We find that $D\,[4] = 0$, $(D-1)^2\,[te^t] = 0$, and $(D^2+4)\,[\sin\, 2t] = 0$. The operator $H(D) = D(D-1)^2(D^2+4)$is an annihilator of $4 + te^t + \sin\, 2t$. The operator corresponding to the left hand side of the ODE is $D^2(D^2+4)$. It follows that

$$D^3(D-1)^2(D^2+4)^2 Y = 0\,.$$

The resulting ODE is homogeneous, with solution $Y(t) = c_1 + c_2 t + c_3 t^2 + c_4 e^t + c_5 te^t + c_6 \cos\, 2t + c_7 \sin\, 2t + c_8 t \cos\, 2t + c_9 t \sin\, 2t$. After examining the homogeneous solution of Problem 16, and eliminating duplicate terms, we have $Y(t) = c_3 t^2 + c_4 e^t + c_5 te^t + c_8 t \cos\, 2t + c_9 t \sin\, 2t$.

22. (18) Observe that $(D-1)\,[e^t] = 0$, $(D+1)^2\,[te^{-t}] = 0$. The function $e^{-t}\sin\, t$ is a solution of a second order ODE with characteristic roots $r = -1 \pm i$. It follows that $(D^2+2D+2)\,[e^{-t}\sin\, t] = 0$. Therefore the operator

$$H(D) = (D-1)(D+1)^2(D^2+2D+2)$$

is an annihilator of $3e^t + 2te^{-t} + e^{-t}\sin\, t$. The operator corresponding to the left hand side of the given ODE is $D^2(D^2+2D+2)$. It follows that

$$D^2(D-1)(D+1)^2(D^2+2D+2)^2 Y = 0\,.$$

The resulting ODE is homogeneous, with solution $Y(t) = c_1 + c_2 t + c_3 e^t + c_4 e^{-t} + c_5 te^{-t} + e^{-t}(c_6 \cos\, t + c_7 \sin\, t) + te^{-t}(c_8 \cos\, t + c_9 \sin\, t\,)$. After examining the homogeneous solution of Problem 18, and eliminating duplicate terms, we have $Y(t) = c_3 e^t + c_4 e^{-t} + c_5 te^{-t} + te^{-t}(c_8 \cos\, t + c_9 \sin\, t\,)$.

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