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2. The characteristic equation for the homogeneous problem is $r^2 + 2r + 5 = 0$, with complex roots $r = -1 \pm 2i$. Hence $y_c(t) = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t$. Since the function $g(t) = 3 \sin 2t$ is not proportional to the solutions of the homogeneous equation, set $Y = A \cos 2t + B \sin 2t$. Substitution into the given differential equation, and comparing the coefficients, results in the system of equations $B - 4A = 3$ and $A + 4B = 0$. Hence $Y = -(12/17) \cos 2t + (3/17) \sin 2t$. The general solution is $y(t) = y_c(t) + Y$.

3. The characteristic equation for the homogeneous problem is $r^2 - r - 2 = 0$, with roots $r = -1, 2$. Hence $y_c(t) = c_1 e^{-t} + c_2 e^{2t}$. Set $Y = At^2 + Bt + C$. Substitution into the given differential equation, and comparing the coefficients, results in the system of equations $-2A = 4$, $-2A - 2B = -2$ and $2A - B - 2C = 0$. Hence $Y = -2t^2 + 3t - 7/2$. The general solution is $y(t) = y_c(t) + Y$.

4. The characteristic equation for the homogeneous problem is $r^2 + r - 6 = 0$, with roots $r = -3, 2$. Hence $y_c(t) = c_1 e^{-3t} + c_2 e^{2t}$. Set $Y = Ae^{3t} + Be^{-2t}$. Substitution into the given differential equation, and comparing the coefficients, results in the system of equations $6A = 12$ and $-4B = 12$. Hence $Y = 2e^{3t} - 3e^{-2t}$. The general solution is $y(t) = y_c(t) + Y$.

5. The characteristic equation for the homogeneous problem is $r^2 - 2r - 3 = 0$, with roots $r = -1, 3$. Hence $y_c(t) = c_1 e^{-t} + c_2 e^{3t}$. Note that the assignment $Y = Ate^{-t}$ is not sufficient to match the coefficients. Try $Y = Ate^{-t} + Bt^2 e^{-t}$. Substitution into the differential equation, and comparing the coefficients, results in the system of equations $-4A + 2B = 0$ and $-8B = -3$. This implies that $Y = (3/16)te^{-t} + (3/8)t^2 e^{-t}$. The general solution is $y(t) = y_c(t) + Y$.

7. The characteristic equation for the homogeneous problem is $r^2 + 9 = 0$, with complex roots $r = \pm 3i$. Hence $y_c(t) = c_1 \cos 3t + c_2 \sin 3t$. To simplify the analysis, set $g_1(t) = 6$ and $g_2(t) = t^2 e^{3t}$. By inspection, we have $Y_1 = 2/3$. Based on the form of g_2, set $Y_2 = Ae^{3t} + Bte^{3t} + Ct^2 e^{3t}$. Substitution into the differential equation, and comparing the coefficients, results in the system of equations $18A + 6B + 2C = 0$, $18B + 12C = 0$, and $18C = 1$. Hence

$$Y_2 = \frac{1}{162} e^{3t} - \frac{1}{27} te^{3t} + \frac{1}{18} t^2 e^{3t}.$$

The general solution is $y(t) = y_c(t) + Y_1 + Y_2$.

9. The characteristic equation for the homogeneous problem is $2r^2 + 3r + 1 = 0$, with roots $r = -1, -1/2$. Hence $y_c(t) = c_1 e^{-t} + c_2 e^{-t/2}$. To simplify the analysis, set $g_1(t) = t^2$ and $g_2(t) = 3 \sin t$. Based on the form of g_1, set $Y_1 = A + Bt + Ct^2$. Substitution into the differential equation, and comparing the coefficients, results in the system of equations $A + 3B + 4C = 0$, $B + 6C = 0$, and $C = 1$. Hence we obtain $Y_1 = 14 - 6t + t^2$. On the other hand, set $Y_2 = D \cos t + E \sin t$. After substitution into the ODE, we find that $D = -9/10$ and $E = -3/10$. The general solution is $y(t) = y_c(t) + Y_1 + Y_2$.

11. The characteristic equation for the homogeneous problem is $r^2 + \omega_0^2 = 0$, with complex roots $r = \pm\,\omega_0\, i$. Hence $y_c(t) = c_1 \cos\,\omega_0\, t + c_2 \sin\,\omega_0\, t$. Since $\omega \neq \omega_0$, set $Y = A\,\cos\,\omega t + B\,\sin\,\omega t$. Substitution into the ODE and comparing the coefficients results in the system of equations $(\omega_0^2 - \omega^2)A = 1$ and $(\omega_0^2 - \omega^2)B = 0$. Hence

$$Y = \frac{1}{\omega_0^2 - \omega^2}\,\cos\,\omega t\,.$$

The general solution is $y(t) = y_c(t) + Y$.

12. From Problem 11, $y_c(t)$ is known. Since $\cos\,\omega_0\, t$ is a solution of the homogeneous problem, set $Y = At\,\cos\,\omega_0\, t + Bt\,\sin\,\omega_0\, t$. Substitution into the given ODE and comparing the coefficients results in $A = 0$ and $B = 1/2\omega_0$. Hence the general solution is $y(t) = c_1 \cos\,\omega_0\, t + c_2 \sin\,\omega_0\, t\, + t \sin\,\omega_0\, t/(2\omega_0\,)$.

14. The characteristic equation for the homogeneous problem is $r^2 - r - 2 = 0$, with roots $r = -1\,,\,2$. Hence $y_c(t) = c_1 e^{-t} + c_2\, e^{2t}$. Based on the form of the right hand side, that is, $\cosh(2t) = (e^{2t} + e^{-2t})/2$, set $Y = At\, e^{2t} + Be^{-2t}$. Substitution into the given ODE and comparing the coefficients results in $A = 1/6$ and $B = 1/8$. Hence the general solution is $y(t) = c_1 e^{-t} + c_2\, e^{2t} + t\, e^{2t}/6 + e^{-2t}/8$.

16. The characteristic equation for the homogeneous problem is $r^2 + 4 = 0$, with roots $r = \pm\, 2i$. Hence $y_c(t) = c_1 \cos\, 2t + c_2 \sin\, 2t$. Set $Y_1 = A + Bt + Ct^2$. Comparing the coefficients of the respective terms, we find that $A = -1/8$, $B = 0$, $C = 1/4$. Now set $Y_2 = D\,e^t$, and obtain $D = 3/5$. Hence the general solution is $y(t) = c_1 \cos\, 2t + c_2 \sin\, 2t - 1/8 + t^2/4 + 3\, e^t/5$. Invoking the initial conditions, we require that $19/40 + c_1 = 0$ and $3/5 + 2c_2 = 2$. Hence $c_1 = -19/40$ and $c_2 = 7/10$.

17. The characteristic equation for the homogeneous problem is $r^2 - 2r + 1 = 0$, with a double root $r = 1$. Hence $y_c(t) = c_1 e^t + c_2 t\, e^t$. Consider $g_1(t) = t\, e^t$. Note that g_1 is a solution of the homogeneous problem. Set $Y_1 = At^2 e^t + Bt^3 e^t$ (the first term is not sufficient for a match). Upon substitution, we obtain $Y_1 = t^3 e^t/6$. By inspection, $Y_2 = 4$. Hence the general solution is $y(t) = c_1 e^t + c_2 t\, e^t + t^3 e^t/6\, + 4$. Invoking the initial conditions, we require that $c_1 + 4 = 1$ and $c_1 + c_2 = 1$. Hence $c_1 = -3$ and $c_2 = 4$.

19. The characteristic equation for the homogeneous problem is $r^2 + 4 = 0$, with roots $r = \pm 2i$. Hence $y_c(t) = c_1 \cos\, 2t + c_2 \sin\, 2t$. Since the function $\sin\, 2t$ is a solution of the homogeneous problem, set $Y = At\,\cos\, 2t + Bt\,\sin\, 2t$. Upon substitution, we obtain $Y = -3t\,\cos\, 2t\,/4$. Hence the general solution is $y(t) = c_1 \cos\, 2t + c_2 \sin\, 2t - 3t\,\cos\, 2t\,/4$. Invoking the initial conditions, we require that $c_1 = 2$ and $2c_2 - (3/4) = -1$. Hence $c_1 = 2$ and $c_2 = -1/8$.

20. The characteristic equation for the homogeneous problem is $r^2 + 2r + 5 = 0$, with complex roots $r = -1 \pm\, 2i$. Hence $y_c(t) = c_1 e^{-t}\cos\, 2t + c_2 e^{-t}\sin\, 2t$. Based on the form of $g(t)$, set $Y = At\, e^{-t}\cos\, 2t + Bt\, e^{-t}\sin\, 2t$. After comparing coefficients, we obtain $Y = t\, e^{-t}\sin\, 2t$. Hence the general solution is $y(t) = c_1 e^{-t}\cos\, 2t + c_2 e^{-t}\sin\, 2t + t\, e^{-t}\sin\, 2t$. Invoking the initial conditions, we require

that $c_1 = 1$ and $-c_1 + 2c_2 = 0$. Hence $c_1 = 1$ and $c_2 = 1/2$.

22.(a) The characteristic equation for the homogeneous problem is $r^2 + 1 = 0$, with complex roots $r = \pm\, i$. Hence $y_c(t) = c_1 \cos t + c_2 \sin t$. Let $g_1(t) = t \sin t$ and $g_2(t) = t$. By inspection, it is easy to see that $Y_2(t) = t$. Based on the form of $g_1(t)$, set $Y_1(t) = At \cos t + Bt \sin t + Ct^2 \cos t + Dt^2 \sin t$.

(b) Substitution into the equation and comparing the coefficients results in $A = 0$, $B = 1/4$, $C = -1/4$, and $D = 0$. Hence $Y(t) = t + t \sin t/4 - t^2 \cos t/4$.

23.(a) The characteristic equation for the homogeneous problem is $r^2 - 5r + 6 = 0$, with roots $r = 2, 3$. Hence $y_c(t) = c_1 e^{2t} + c_2 e^{3t}$. Consider $g_1(t) = e^{2t}(3t + 4)\sin t$, and $g_2(t) = e^t \cos 2t$. Based on the form of these functions on the right hand side of the ODE, set $Y_2(t) = e^t(A_1 \cos 2t + A_2 \sin 2t)$ and $Y_1(t) = (B_1 + B_2 t)e^{2t} \sin t + (C_1 + C_2 t)e^{2t} \cos t$.

(b) Substitution into the equation and comparing the coefficients results in

$$Y(t) = -\frac{1}{20}(e^t \cos 2t + 3e^t \sin 2t) + \frac{3}{2}te^{2t}(\cos t - \sin t) + e^{2t}(\frac{1}{2}\cos t - 5\sin t).$$

25.(a) We obtain the double characteristic root $r = 2$. Hence $y_c(t) = c_1 e^{2t} + c_2 te^{2t}$. Consider the functions $g_1(t) = 2t^2$, $g_2(t) = 4te^{2t}$, and $g_3(t) = t \sin 2t$. The corresponding forms of the respective parts of the particular solution are $Y_1(t) = A_0 + A_1 t + A_2 t^2$, $Y_2(t) = e^{2t}(B_2 t^2 + B_3 t^3)$, and $Y_3(t) = t(C_1 \cos 2t + C_2 \sin 2t) + (D_1 \cos 2t + D_2 \sin 2t)$.

(b) Substitution into the equation and comparing the coefficients results in

$$Y(t) = \frac{1}{4}(3 + 4t + 2t^2) + \frac{2}{3}t^3 e^{2t} + \frac{1}{8}t \cos 2t + \frac{1}{16}(\cos 2t - \sin 2t).$$

26.(a) The homogeneous solution is $y_c(t) = c_1 \cos 2t + c_2 \sin 2t$. Since $\cos 2t$ and $\sin 2t$ are both solutions of the homogeneous equation, set

$$Y(t) = t(A_0 + A_1 t + A_2 t^2)\cos 2t + t(B_0 + B_1 t + B_2 t^2)\sin 2t.$$

(b) Substitution into the equation and comparing the coefficients results in

$$Y(t) = (\frac{13}{32}t - \frac{1}{12}t^3)\cos 2t + \frac{1}{16}(28t + 13t^2)\sin 2t.$$

27.(a) The homogeneous solution is $y_c(t) = c_1 e^{-t} + c_2 te^{-2t}$. None of the functions on the right hand side are solutions of the homogenous equation. In order to include all possible combinations of the derivatives, consider

$$\begin{aligned} Y(t) = {} & e^t(A_0 + A_1 t + A_2 t^2)\cos 2t + e^t(B_0 + B_1 t + B_2 t^2)\sin 2t + \\ & + e^{-t}(C_1 \cos t + C_2 \sin t) + De^t. \end{aligned}$$

(b) Substitution into the differential equation and comparing the coefficients results in

$$\begin{aligned}Y(t) &= e^t(A_0 + A_1 t + A_2 t^2)\cos 2t + e^t(B_0 + B_1 t + B_2 t^2)\sin 2t \\ &+ e^{-t}(-\frac{3}{2}\cos t + \frac{3}{2}\sin t) + 2e^t/3,\end{aligned}$$

in which $A_0 = -4105/35152$, $A_1 = 73/676$, $A_2 = -5/52$, $B_0 = -1233/35152$, $B_1 = 10/169$, $B_2 = 1/52$.

28.(a) The homogeneous solution is $y_c(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$. None of the terms on the right hand side are solutions of the homogenous equation. In order to include the appropriate combinations of derivatives, consider

$$\begin{aligned}Y(t) &= e^{-t}(A_1 t + A_2 t^2)\cos 2t + e^{-t}(B_1 t + B_2 t^2)\sin 2t + \\ &+ e^{-2t}(C_0 + C_1 t)\cos 2t + e^{-2t}(D_0 + D_1 t)\sin 2t.\end{aligned}$$

(b) Substitution into the differential equation and comparing the coefficients results in

$$\begin{aligned}Y(t) &= \frac{3}{16}te^{-t}\cos 2t + \frac{3}{8}t^2 e^{-t}\sin 2t \\ &- \frac{1}{25}e^{-2t}(7 + 10t)\cos 2t + \frac{1}{25}e^{-2t}(1 + 5t)\sin 2t\,.\end{aligned}$$

30. The homogeneous solution is $y_c(t) = c_1\cos\lambda t + c_2\sin\lambda t$. Since the differential operator does not contain a first derivative (and $\lambda \neq m\pi$), we can set

$$Y(t) = \sum_{m=1}^{N} C_m \sin m\pi t\,.$$

Substitution into the differential equation yields

$$-\sum_{m=1}^{N} m^2\pi^2 C_m \sin m\pi t + \lambda^2\sum_{m=1}^{N} C_m \sin m\pi t = \sum_{m=1}^{N} a_m \sin m\pi t\,.$$

Equating coefficients of the individual terms, we obtain

$$C_m = \frac{a_m}{\lambda^2 - m^2\pi^2}\,,\quad m = 1, 2\ldots N.$$

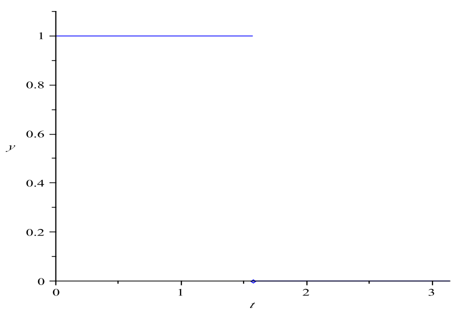
32. The homogeneous solution is $y_c(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$. The input function is independent of the homogeneous solutions, on any interval. Since the right hand side is piecewise constant, it follows by inspection that

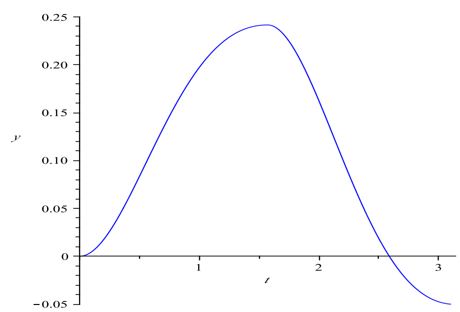
$$Y(t) = \begin{cases} 1/5\,, & 0 \le t \le \pi/2 \\ 0, & t > \pi/2 \end{cases}\,.$$

For $0 \le t \le \pi/2$, the general solution is $y(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t + 1/5$. Invoking the initial conditions $y(0) = y'(0) = 0$, we require that $c_1 = -1/5$, and that $c_2 = -1/10$. Hence

$$y(t) = \frac{1}{5} - \frac{1}{10}(2e^{-t}\cos 2t + e^{-t}\sin 2t)$$

on the interval $0 \le t \le \pi/2$. We now have the values $y(\pi/2) = (1 + e^{-\pi/2})/5$, and $y'(\pi/2) = 0$. For $t > \pi/2$, the general solution is $y(t) = d_1 e^{-t}\cos 2t + d_2 e^{-t}\sin 2t$. It follows that $y(\pi/2) = -e^{-\pi/2}d_1$ and $y'(\pi/2) = e^{-\pi/2}d_1 - 2e^{-\pi/2}d_2$. Since the solution is continuously differentiable, we require that $-e^{-\pi/2}d_1 = (1 + e^{-\pi/2})/5$ and $e^{-\pi/2}d_1 - 2e^{-\pi/2}d_2 = 0$. Solving for the coefficients, $d_1 = 2d_2 = -(e^{\pi/2} + 1)/5$.





34. Since $a, b, c > 0$, the roots of the characteristic equation have negative real parts. That is, $r = \alpha \pm \beta i$, where $\alpha < 0$. Hence the homogeneous solution is

$$y_c(t) = c_1 e^{\alpha t}\cos \beta t + c_2 e^{\alpha t}\sin \beta t.$$

If $g(t) = d$, then the general solution is

$$y(t) = d/c + c_1 e^{\alpha t}\cos \beta t + c_2 e^{\alpha t}\sin \beta t.$$

Since $\alpha < 0$, $y(t) \to d/c$ as $t \to \infty$. If $c = 0$, then the characteristic roots are $r = 0$ and $r = -b/a$. The ODE becomes $ay'' + by' = d$. Integrating both sides, we find that $ay' + by = d\,t + c_1$. The general solution can be expressed as

$$y(t) = d\,t/b + c_1 + c_2 e^{-bt/a}.$$

In this case, the solution grows without bound. If $b = 0$, also, then the differential equation can be written as $y'' = d/a$, which has general solution $y(t) = d\,t^2/2a + c_1 + c_2$. Hence the assertion is true only if the coefficients are positive.

35.(a) Since D is a linear operator, $D^2y + bDy + cy = D^2y - (r_1 + r_2)Dy + r_1 r_2 y = D^2y - r_2Dy - r_1Dy + r_1r_2y = D(Dy - r_2y) - r_1(Dy - r_2y) = (D - r_1)(D - r_2)y$.

(b) Let $u = (D - r_2)y$. Then the ODE (i) can be written as $(D - r_1)u = g(t)$, that is, $u' - r_1u = g(t)$. The latter is a linear first order equation in u. Its general solution is

$$u(t) = e^{r_1 t}\int_{t_0}^{t} e^{-r_1\tau}g(\tau)d\tau + c_1 e^{r_1 t}.$$

From above, we have $y' - r_2 y = u(t)$. This equation is also a first order ODE. Hence the general solution of the original second order equation is

$$y(t) = e^{r_2 t} \int_{t_0}^{t} e^{-r_2 \tau} u(\tau) d\tau + c_2 e^{r_2 t} .$$

Note that the solution $y(t)$ contains two arbitrary constants.

37. Note that $(2D^2 + 3D + 1)y = (2D + 1)(D + 1)y$. Let $u = (D + 1)y$, and solve the ODE $2u' + u = t^2 + 3\sin t$. This equation is a linear first order ODE, with solution

$$u(t) = e^{-t/2} \int_{t_0}^{t} e^{\tau/2} \left[\tau^2/2 + \frac{3}{2} \sin \tau \right] d\tau + c\, e^{-t/2} =$$

$$= t^2 - 4t + 8 - \frac{6}{5} \cos t + \frac{3}{5} \sin t + c\, e^{-t/2}.$$

Now consider the ODE $y' + y = u(t)$. The general solution of this first order ODE is

$$y(t) = e^{-t} \int_{t_0}^{t} e^{\tau} u(\tau) d\tau + c_2 e^{-t} ,$$

in which $u(t)$ is given above. Substituting for $u(t)$ and performing the integration,

$$y(t) = t^2 - 6t + 14 - \frac{9}{10} \cos t - \frac{3}{10} \sin t + c_1 e^{-t/2} + c_2 e^{-t}.$$

38. We have $(D^2 + 2D + 1)y = (D + 1)(D + 1)y$. Let $u = (D + 1)y$, and consider the ODE $u' + u = 2e^{-t}$. The general solution is $u(t) = 2t\, e^{-t} + c\, e^{-t}$. We therefore have the first order equation $u' + u = 2t\, e^{-t} + c_1 e^{-t}$. The general solution of the latter differential equation is

$$y(t) = e^{-t} \int_{t_0}^{t} [2\tau + c_1]\, d\tau + c_2 e^{-t} = e^{-t}(t^2 + c_1 t + c_2).$$

39. We have $(D^2 + 2D)y = D(D + 2)y$. Let $u = (D + 2)y$, and consider the equation $u' = 3 + 4\sin 2t$. Direct integration results in $u(t) = 3t - 2\cos 2t + c$. The problem is reduced to solving the ODE $y' + 2y = 3t - 2\cos 2t + c$. The general solution of this first order differential equation is

$$y(t) = e^{-2t} \int_{t_0}^{t} e^{2\tau} [3\tau - 2\cos 2\tau + c]\, d\tau + c_2 e^{-2t} =$$

$$= \frac{3}{2} t - \frac{1}{2} (\cos 2t + \sin 2t) + c_1 + c_2 e^{-2t}.$$

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