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3.2

1.

$$W(e^{2t}, e^{-3t/2}) = \begin{vmatrix} e^{2t} & e^{-3t/2} \\ 2e^{2t} & -\frac{3}{2}e^{-3t/2} \end{vmatrix} = -\frac{7}{2}e^{t/2}.$$

3.

$$W(e^{-2t}, t\,e^{-2t}) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix} = e^{-4t}\,.$$

5.

$$W(e^t \sin t\,, e^t \cos t) = \begin{vmatrix} e^t \sin t & e^t \cos t \\ e^t(\sin t + \cos t) & e^t(\cos t - \sin t) \end{vmatrix} = -e^{2t}\,.$$

6.

$$W(\cos^2\theta\,, 1+\cos 2\theta) = \begin{vmatrix} \cos^2\theta & 1+\cos 2\theta \\ -2\sin\theta\cos\theta & -2\sin 2\theta \end{vmatrix} = 0\,.$$

7. Write the equation as $y'' + (3/t)y' = 1$. $p(t) = 3/t$ is continuous for all $t > 0$. Since $t_0 > 0$, the IVP has a unique solution for all $t > 0$.

9. Write the equation as $y'' + (3/(t-4))y' + (4/t(t-4))y = 2/t(t-4)$. The coefficients are not continuous at $t = 0$ and $t = 4$. Since $t_0 \in (0\,,4)$, the largest interval is $0 < t < 4$.

10. The coefficient $3\ln|t|$ is discontinuous at $t = 0$. Since $t_0 > 0$, the largest interval of existence is $0 < t < \infty$.

11. Write the equation as $y'' + (x/(x-3))y' + (\ln|x|/(x-3))y = 0$. The coefficients are discontinuous at $x = 0$ and $x = 3$. Since $x_0 \in (0\,,3)$, the largest interval is $0 < x < 3$.

13. $y_1'' = 2$. We see that $t^2(2) - 2(t^2) = 0$. $y_2'' = 2\,t^{-3}$, with $t^2(y_2'') - 2(y_2) = 0$. Let $y_3 = c_1t^2 + c_2t^{-1}$, then $y_3'' = 2c_1 + 2c_2t^{-3}$. It is evident that y_3 is also a solution.

16. No. Substituting $y = \sin(t^2)$ into the differential equation,

$$-4t^2\sin(t^2) + 2\cos(t^2) + 2t\,\cos(t^2)p(t) + \sin(t^2)q(t) = 0\,.$$

At $t = 0$, this equation becomes $2 = 0$ (if we suppose that $p(t)$ and $q(t)$ are continuous), which is impossible.

17. $W(e^{2t}, g(t)) = e^{2t}g'(t) - 2e^{2t}g(t) = 3e^{4t}$. Dividing both sides by e^{2t}, we find that g must satisfy the ODE $g' - 2g = 3e^{2t}$. Hence $g(t) = 3t\,e^{2t} + c\,e^{2t}$.

19. $W(f\,,g) = fg' - f'g$. Also, $W(u\,,v) = W(2f - g\,,f + 2g)$. Upon evaluation, $W(u\,,v) = 5fg' - 5f'g = 5W(f\,,g)$.

20. $W(f\,,g) = fg' - f'g = t\,\cos t - \sin t$, and $W(u\,,v) = -4fg' + 4f'g$. Hence $W(u\,,v) = -4t\,\cos t + 4\sin t$.

21. We compute

$$W(a_1y_1 + a_2y_2, b_1y_1 + b_2y_2) = \begin{vmatrix} a_1y_1 + a_2y_2 & b_1y_1 + b_2y_2 \\ a_1y_1' + a_2y_2' & b_1y_1' + b_2y_2' \end{vmatrix} =$$

$$= (a_1y_1 + a_2y_2)(b_1y_1' + b_2y_2') - (b_1y_1 + b_2y_2)(a_1y_1' + a_2y_2') =$$

$$= a_1b_2(y_1y_2' - y_1'y_2) - a_2b_1(y_1y_2' - y_1'y_2) = (a_1b_2 - a_2b_1)W(y_1, y_2).$$

This now readily shows that y_3 and y_4 form a fundamental set of solutions if and only if $a_1b_2 - a_2b_1 \neq 0$.

23. The general solution is $y = c_1e^{-3t} + c_2e^{-t}$. $W(e^{-3t}, e^{-t}) = 2e^{-4t}$, and hence the exponentials form a fundamental set of solutions. On the other hand, the fundamental solutions must also satisfy the conditions $y_1(1) = 1$, $y_1'(1) = 0$; $y_2(1) = 0$, $y_2'(1) = 1$. For y_1, the initial conditions require $c_1 + c_2 = e$, $-3c_1 - c_2 = 0$. The

coefficients are $c_1 = -e^3/2$, $c_2 = 3e/2$. For the solution y_2, the initial conditions require $c_1 + c_2 = 0$, $-3c_1 - c_2 = e$. The coefficients are $c_1 = -e^3/2$, $c_2 = e/2$. Hence the fundamental solutions are

$$y_1 = -\frac{1}{2}e^{-3(t-1)} + \frac{3}{2}e^{-(t-1)} \quad \text{and} \quad y_2 = -\frac{1}{2}e^{-3(t-1)} + \frac{1}{2}e^{-(t-1)}.$$

24. Yes. $y_1'' = -4\cos 2t$; $y_2'' = -4\sin 2t$. $W(\cos 2t, \sin 2t) = 2$.

25. Clearly, $y_1 = e^t$ is a solution. $y_2' = (1+t)e^t$, $y_2'' = (2+t)e^t$. Substitution into the ODE results in $(2+t)e^t - 2(1+t)e^t + t\,e^t = 0$. Furthermore, $W(e^t, te^t) = e^{2t}$. Hence the solutions form a fundamental set of solutions.

27. Clearly, $y_1 = x$ is a solution. $y_2' = \cos x$, $y_2'' = -\sin x$. Substitution into the ODE results in $(1 - x\cot x)(-\sin x) - x(\cos x) + \sin x = 0$. We can compute that $W(y_1, y_2) = x\cos x - \sin x$, which is nonzero for $0 < x < \pi$. Hence $\{x, \sin x\}$ is a fundamental set of solutions.

30. Writing the equation in standard form, we find that $P(t) = \sin t/\cos t$. Hence the Wronskian is $W(t) = c\,e^{-\int(\sin t/\cos t)\,dt} = c\,e^{\ln|\cos t|} = c\cos t$, in which c is some constant.

31. After writing the equation in standard form, we have $P(x) = 1/x$. The Wronskian is $W(x) = c\,e^{-\int(1/x)\,dx} = c\,e^{-\ln|x|} = c/x$, in which c is some constant.

32. Writing the equation in standard form, we find that $P(x) = -2x/(1-x^2)$. The Wronskian is $W(x) = c\,e^{-\int -2x/(1-x^2)\,dx} = c\,e^{-\ln\left|1-x^2\right|} = c/(1-x^2)$, in which c is some constant.

33. Rewrite the equation as $p(t)y'' + p'(t)y' + q(t)y = 0$. After writing the equation in standard form, we have $P(t) = p'(t)/p(t)$. Hence the Wronskian is

$$W(t) = c\,e^{-\int p'(t)/p(t)\,dt} = c\,e^{-\ln p(t)} = c/p(t).$$

35. The Wronskian associated with the solutions of the differential equation is given by $W(t) = c\,e^{-\int -2/t^2\,dt} = c\,e^{-2/t}$. Since $W(2) = 3$, it follows that for the hypothesized set of solutions, $c = 3\,e$. Hence $W(4) = 3\sqrt{e}$.

36. For the given differential equation, the Wronskian satisfies the first order differential equation $W' + p(t)W = 0$. Given that W is constant, it is necessary that $p(t) \equiv 0$.

37. Direct calculation shows that $W(fg, fh) = (fg)(fh)' - (fg)'(fh) = (fg)(f'h + fh') - (f'g + fg')(fh) = f^2\,W(g, h)$.

39. Since y_1 and y_2 are solutions, they are differentiable. The hypothesis can thus be restated as $y_1'(t_0) = y_2'(t_0) = 0$ at some point t_0 in the interval of definition.

This implies that $W(y_1\,,y_2)(t_0) = 0$. But $W(y_1\,,y_2)(t_0) = c\,e^{-\int p(t)dt}$, which cannot be equal to zero, unless $c = 0$. Hence $W(y_1\,,y_2) \equiv 0$, which is ruled out for a fundamental set of solutions.

42. $P = 1$, $Q = x$, $R = 1$. We have $P'' - Q' + R = 0$. The equation is exact. Note that $(y')' + (xy)' = 0$. Hence $y' + xy = c_1$. This equation is linear, with integrating factor $\mu = e^{x^2/2}$. Therefore the general solution is

$$y(x) = c_1 e^{-x^2/2}\int_{x_0}^{x} e^{u^2/2}du + c_2 e^{-x^2/2}.$$

43. $P = 1$, $Q = 3x^2$, $R = x$. Note that $P'' - Q' + R = -5x$, and therefore the differential equation is not exact.

45. $P = x^2$, $Q = x$, $R = -1$. We have $P'' - Q' + R = 0$. The equation is exact. Write the equation as $(x^2 y')' - (xy)' = 0$. After integration, we conclude that $x^2 y' - xy = c$. Divide both sides of the differential equation by x^2. The resulting equation is linear, with integrating factor $\mu = 1/x$. Hence $(y/x)' = c\,x^{-3}$. The solution is $y(t) = c_1 x^{-1} + c_2 x$.

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