[redacted]

1. Let $Q(t)$ be the quantity of dye in the tank. We know that

$$\frac{dQ}{dt} = \text{rate in } - \text{ rate out.}$$

Here, fresh water is flowing in. Therefore, no dye is coming in. The dye is flowing out at the rate of $(Q/200)g/l \cdot 2l/min = Q/100$ l/min. Therefore,

$$\frac{dQ}{dt} = -\frac{Q}{100}.$$

The solution of this equation is $Q(t) = Ce^{-t/100}$. Since $Q(0) = 200$ grams, $C = 200$. We need to find the time T when the amount of dye present is 1% of what it is initially. That is, we need to find the time T when $Q(T) = 2$ grams. Solving the equation $2 = 200e^{-t/100}$, we conclude that $T = 100\ln(100)$ minutes.

2. Let $Q(t)$ be the quantity of salt in the tank. We know that

$$\frac{dQ}{dt} = \text{rate in } - \text{ rate out.}$$

Here, water containing γ g/liter of salt is flowing in at a rate of 2 liters/minute. The salt is flowing out at the rate of $(Q/120)g/l \cdot 2l/min = Q/60$ l/min. Therefore,

$$\frac{dQ}{dt} = 2\gamma - \frac{Q}{60}.$$

The solution of this equation is $Q(t) = 120\gamma + Ce^{-t/60}$. Since $Q(0) = 0$ grams, $C = -120\gamma$. Therefore, $Q(t) = 120\gamma[1 - e^{-t/60}]$. As $t \to \infty$, $Q(t) \to 120\gamma$.

3. Let $Q(t)$ be the quantity of salt in the tank. We know that

$$\frac{dQ}{dt} = \text{rate in } - \text{ rate out}.$$

Here, water containing 1/2 lb/gallon of salt is flowing in at a rate of 2 gal/minute. The salt is flowing out at the rate of $(Q/100)lb/gal \cdot 2gal/min = Q/50$ gal/min. Therefore,

$$\frac{dQ}{dt} = 1 - \frac{Q}{50}.$$

The solution of this equation is $Q(t) = 50 + Ce^{-t/50}$. Since $Q(0) = 0$ grams, $C = -50$. Therefore, $Q(t) = 50[1 - e^{-t/50}]$ for $0 \leq t \leq 10$ minutes. After 10 minutes, the amount of salt in the tank is $Q(10) = 50[1 - e^{-1/5}] \approx 9.06$ lbs. Starting at that time (and resetting the time variable), the new equation for dQ/dt is given by

$$\frac{dQ}{dt} = -\frac{Q}{50},$$

since fresh water is being added. The solution of this equation is $Q(t) = Ce^{-t/50}$. Since we are now starting with 9.06 lbs of salt, $Q(0) = 9.06 = C$. Therefore, $Q(t) = 9.06e^{-t/50}$. After 10 minutes, $Q(10) = 9.06e^{-1/5} \cong 7.42$ lbs.

4. Let $Q(t)$ be the quantity of salt in the tank. We know that

$$\frac{dQ}{dt} = \text{rate in } - \text{ rate out}.$$

Here, water containing 1 lb/gallon of salt is flowing in at a rate of 3 gal/minute. The salt is flowing out at the rate of $(Q/(200+t))lb/gal \cdot 2gal/min = 2Q/(200+t)$ lb/min. Therefore,

$$\frac{dQ}{dt} = 3 - \frac{2Q}{200+t}.$$

This is a linear equation with integrating factor $\mu(t) = (200+t)^2$. The solution of this equation is $Q(t) = 200 + t + C(200+t)^{-2}$. Since $Q(0) = 100$ lbs, $C = -4,000,000$. Therefore, $Q(t) = 200 + t - 4,000,000/(200+t)^2$. Since the tank has a net gain of 1 gallon of water every minute, the tank will reach its capacity after 300 minutes. When $t = 300$, we see that $Q(300) = 484$ lbs. Therefore, the concentration of salt when it is on the point of overflowing is 121/125 lbs/gallon. The concentration of salt is given by $Q(t)/(200+t)$ (since t gallons of water are added every t minutes). Using the equation for Q above, we see that if the tank had infinite capacity, the concentration would approach 1 as $t \to \infty$.

5.

(a) Let $Q(t)$ be the quantity of salt in the tank. We know that

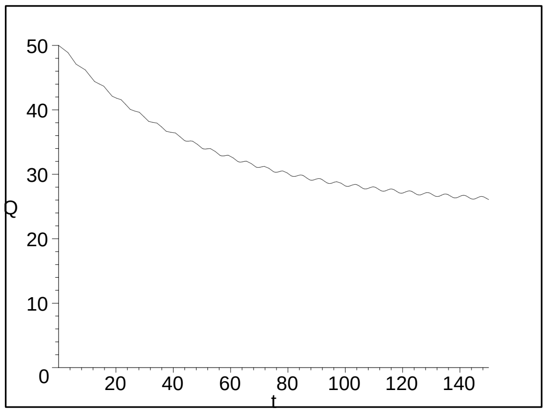
$$\frac{dQ}{dt} = \text{rate in } - \text{ rate out}.$$

Here, water containing $\frac{1}{4}\left(1+\frac{1}{2}\sin t\right)$ oz/gallon of salt is flowing in at a rate of 2 gal/minute. The salt is flowing out at the rate of $Q/100oz/gal \cdot 2gal/min = Q/50$ oz/min. Therefore,

$$\frac{dQ}{dt} = \frac{1}{2} + \frac{1}{4}\sin t - \frac{Q}{50}.$$

This is a linear equation with integrating factor $\mu(t) = e^{t/50}$. The solution of this equation is $Q(t) = (12.5\sin t - 625\cos t + 63150e^{-t/50})/2501 + C$. The initial condition, $Q(0) = 50$ oz implies $C = 25$. Therefore, $Q(t) = 25 + (12.5\sin t - 625\cos t + 63150e^{-t/50})/2501$.

(b)



(c) The amount of salt approaches a steady state, which is an oscillation of amplitude 1/4 about a level of 25 oz.

6.

(a) Using the Principle of Conservation of Energy, we know that the kinetic energy of a particle after it has fallen from a height h is equal to its potential energy at a height t. Therefore, $mv^2/2 = mgh$. Solving this equation for v, we have $v = \sqrt{2gh}$.

(b) The volumetric outflow rate is (outflow cross-sectional area) $\times$ (outflow velocity): $\alpha a\sqrt{2gh}$. The volume of water in the tank is

$$V(h) = \int_0^h A(u)\,du$$

where $A(u)$ is the cross-sectional area of the tank at height u. By the chain rule,

$$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt} = A(h)\frac{dh}{dt}.$$

Therefore,

$$\frac{dV}{dt} = A(h)\frac{dh}{dt} = -\alpha a\sqrt{2gh}.$$

(c) The cross-sectional area of the cylinder is $A(h) = \pi(1)^2 = \pi$. The outflow cross-sectional area is $a = \pi(.1)^2 = .01\pi$. From part (a), we take $\alpha = 0.6$ for water. Then by part (b), we have

$$\pi \frac{dh}{dt} = -0.006\pi\sqrt{2gh}.$$

This is a separable equation with solution $h(t) = 0.000018gt^2 - 0.006\sqrt{2gh(0)}t + h(0)$. Setting $h(0) = 3$ and $g = 9.8$, we have $h(t) = 0.0001764t^2 - 0.046t + 3$. Then $h(t) = 0$ implies $t \approx 130.4$ seconds.

7.

(a) The equation for S is

$$\frac{dS}{dt} = rS$$

with an initial condition $S(0) = S_0$. The solution of the equation is $S(t) = S_0e^{rt}$. We want to find the time T such that $S(T) = 2S_0$. Our equation becomes $2S_0 = S_0e^{rT}$. Dividing by S_0 and applying the logarithmic function to our equation, we have $rT = \ln(2)$. That is, $T = \ln(2)/r$.

(b) If $r = .07$, then $T = \ln(2)/.07 \cong 9.90$ years.

(c) By part (a), we also know that $r = \ln(2)T$ where T is the doubling time. If we want the investment to double in $T = 8$ years, then we need $r = \ln(2)/8 \cong 8.66\%$.

8.

(a) The equation for S is given by

$$\frac{dS}{dt} = rS + k.$$

This is a linear equation with solution $S(t) = \frac{k}{r}[e^{rt} - 1]$.

(b) Using the function in part (a), we need to find k so that $S(40) = 1,000,000$ assuming $r = .075$. That is, we need to solve

$$1,000,000 = \frac{k}{.075}[e^{.075 \cdot 40} - 1].$$

The solution of this equation is $k \cong 3930$.

(c) Now we assume that $k = 2000$ and want to find r. Our equation becomes

$$1,000,000 = \frac{2000}{r}[e^{40r} - 1].$$

The solution of this equation is approximately 9.77%.

9.

(a) Let $S(t)$ be the balance due on the loan at time t. To determine the maximum amount the buyer can afford to borrow, we will assume that the buyer will pay \$800 per month. Then

$$\frac{dS}{dt} = .09S - 12(800).$$

The solution is given by equation (18), $S(t) = S_0 e^{.09t} - 106,667(e^{.09t} - 1)$. If the term of the mortgage is 20 years, then $S(20) = 0$. Therefore, $0 = S_0 e^{.09(20)} - 106,667(e^{.09(20)} - 1)$ which implies $S_0 \approx \$89,035$.

(b) Since the homeowner pays \$800 per month for 20 years, he ends up paying a total of \$192,000 for the house. Since the house loan was \$89,035, the rest of the amount was interest payments. Therefore, the amount of interest was approximately \$102,965.

10.

(a) Let $S(t)$ be the balance due on the loan at time t. Taking into account that t is measured in years, we rewrite the monthly payment as $800(1 + t/10)$ where t is now in years. The equation for S is given by

$$\frac{dS}{dt} = .09S - 12(800)(1 + t/10).$$

This is a linear equation. Its solution is $S(t) = 225185 + 10667t + ce^{.09t}$. The initial condition $S(0) = 100,000$ implies $c = -125185$. Therefore, the particular solution is $S(t) = 225185 + 10667t - 125185e^{.09t}$. To find when the loan will be paid, we just need to solve $S(t) = 0$. Solving this equation, we conclude that the loan will be paid off in 11.28 years (135.36 months).

(b) From part (a), we know the general solution is given by $S(t) = 225185 + 10667t + ce^{.09t}$. Now we want to find c such that $S(20) = 0$. The solution of this equation is $c = -72486$. Therefore, the solution of the equation will be $S(t) = 225185 + 10667 - 72846e^{.09t}$. Therefore, $S(0) = 225185 - 72846 = 152699$.

11.

(a) If S_0 is the initial balance, then the balance after one month is

$$\begin{aligned} S_1 &= \text{ initial balance + interest - monthly payment} \\ &= S_0 + rS_0 - k. \end{aligned}$$

Similarly,

$$\begin{aligned} S_2 &= S_1 + rS_1 - k \\ &= (1 + r)S_1 - k. \end{aligned}$$

In general,

$$S_n = (1 + r)S_{n-1} - k.$$

(b) $R = 1 + r$ implies $S_n = RS_{n-1} - k$. Therefore,

$$\begin{aligned} S_1 &= RS_0 - k \\ S_2 &= RS_1 - k = R[RS_0 - k] - k = R^2 S_0 - (R+1)k \\ S_3 &= RS_2 - k = R[R^2 S_0 - (R+1)k] - k = R^3 S_0 - (R^2 + R + 1)k. \end{aligned}$$

(c) We check the base case, $n = 1$. We see that

$$S_1 = RS_0 - k = RS_0 - \left(\frac{R-1}{R-1}\right) k,$$

which implies that that the condition is satisfied for $n = 1$. We assume that

$$S_n = R^n S_0 - \frac{R^n - 1}{R - 1} k$$

to show that

$$S_{n+1} = R^{n+1} S_0 - \frac{R^{n+1} - 1}{R - 1} k.$$

We see that

$$\begin{aligned} S_{n+1} &= RS_n - k \\ &= R\left[R^n S_0 - \frac{R^n - 1}{R - 1} k\right] - k \\ &= R^{n+1} S_0 - \left(\frac{R^{n+1} - R}{R - 1}\right) k - k \\ &= R^{n+1} S_0 - \left(\frac{R^{n+1} - R}{R - 1}\right) k - \left(\frac{R-1}{R-1}\right) k \\ &= R^{n+1} S_0 - \left(\frac{R^{n+1} - R + R - 1}{R - 1}\right) k \\ &= R^{n+1} S_0 - \left(\frac{R^{n+1} - 1}{R - 1}\right) k. \end{aligned}$$

(d) We are assuming that $S_0 = 20,000$ and $r = .08/12$. We need to find k such that $S_{48} = 0$. Our equation becomes

$$S_{48} = R^{48} S_0 - \left(\frac{R^{48} - 1}{R - 1}\right) k = 0.$$

Therefore,

$$\left(\frac{(1 + .08/12)^{48} - 1}{.08/12}\right) k = \left(1 + \frac{.08}{12}\right)^{48} \cdot 20,000,$$

which implies $k \approx 488.26$, which is very close to the result in example 2.

12.

(a) The general solution is $Q(t) = Q_0 e^{-rt}$. If the half-life is 5730, then $Q_0/2 = Q_0 e^{-5730r}$ implies $-5730r = \ln(1/2)$. Therefore, $r = 1.2097 \times 10^{-4}$ per year.

(b) Therefore, $Q(t) = Q_0 e^{-1.2097 \times 10^{-4} t}$.

(c) Given that $Q(T) = Q_0/5$, we have the equation $1/5 = e^{-1.2097 \times 10^{-4} T}$. Solving for T, we have $T = 13,304.65$ years.

13. Let $P(t)$ be the population of mosquitoes at any time t, measured in days. Then

$$\frac{dP}{dt} = rP - 20,000.$$

The solution of this linear equation is $P(t) = P_0 e^{rt} - \frac{20,000}{r}(e^{rt} - 1)$. In the absence of predators, the equation is $dP_1/dt = rP_1$. The solution of this equation is $P_1(t) = P_0 e^{rt}$. Since the population doubles after 7 days, we see that $2P_0 = P_0 e^{7r}$. Therefore, $r = \ln(2)/7 = .09902$ per day. Therefore, the population of mosquitoes at any time t is given by $P(t) = 200,000 e^{.099t} - 201,997(e^{.099t} - 1) = 201,997 - 1997 e^{.099t}$.

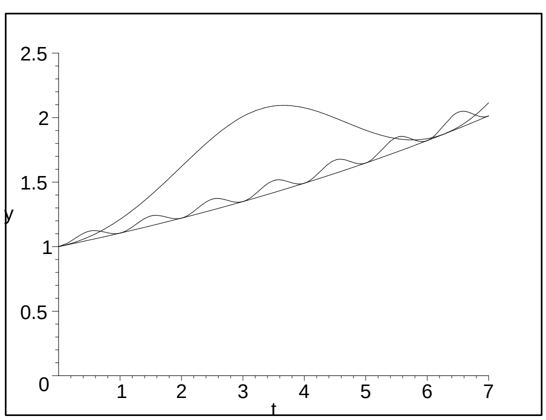
14.

(a) The solution of this separable equation is given by $y(t) = \exp[2/10 + t/10 - 2\cos(t)/10]$. The doubling-time is found by solving the equation $2 = \exp[2/10 + t/10 - 2\cos(t)/10]$. The solution of this equation is given by $\tau \approx 2.9632$.

(b) The differential equation will be $dy/dt = y/10$ with solution $y(t) = y(0)e^{t/10}$. The doubling time is found by setting $y(t) = 2y(0)$. In this case, the doubling time is $\tau \approx 6.9315$.

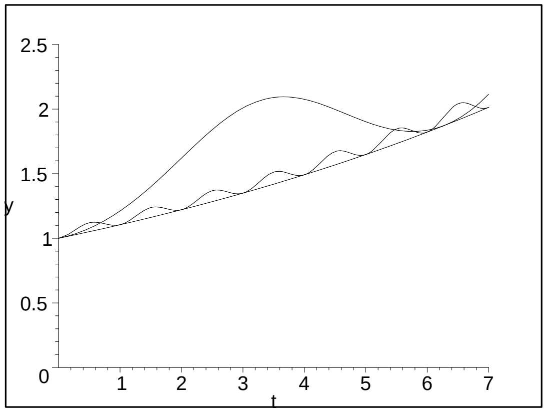
(c) Consider the differential equation $dy/dt = (0.5 + \sin(2\pi t))y/5$. This equation is separable with solution $y(t) = \exp[(1 + \pi t - \cos(2\pi t))/(10\pi)]$. The doubling time is found by setting $y(t) = 2$. The solution is given by 6.9167.

(d)



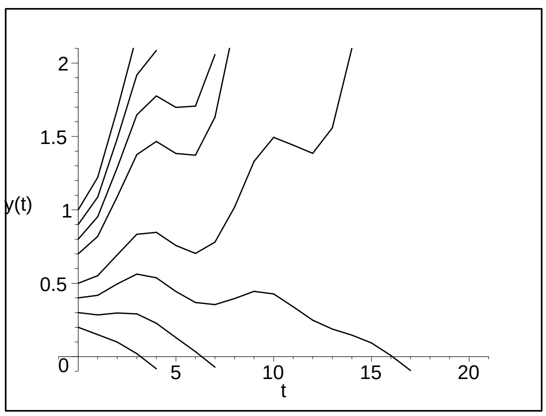
15.

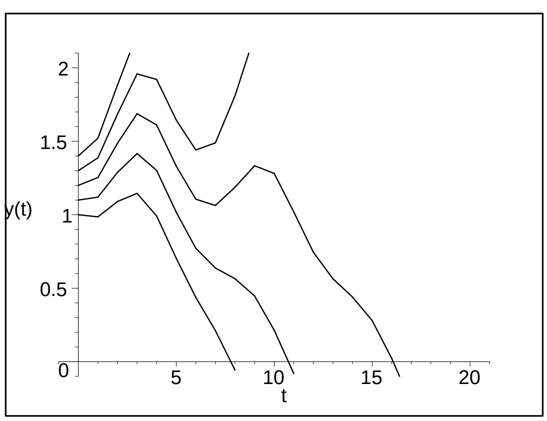
(a)



(b) Based on the graph, we estimate that $y_c \approx 0.83$.

(c) We sketch the graphs below for $k = 1/10$ and $k = 3/10$, respectively. Based on these graphs, we estimate that $y_c(1/10) \approx .41$ and $y_c(3/10) \approx 1.24$.





(d) From our results from above, we conclude that y_c is a linear function of k.

16. Let $T(t)$ be the temperature of the coffee at time t. The governing equation is given by

$$\frac{dT}{dt} = -k(T - 70).$$

This is a linear equation with solution $T(t) = 70 + ce^{-kt}$. The initial condition $T(0) = 200$ implies $c = 130$. Therefore, $T(t) = 70 + 130e^{-kt}$. Using the fact that $T(1) = 190$, we see that $190 = 70 + 130e^{-k}$ which implies $k = -\ln(12/13) \cong .08$ per minute. To find when the temperature reaches 150 degrees, we just need to solve $T(t) = 70 + 130e^{-0.08t} = 150$. The solution of this equation is $t = -\ln(80/130)/.08 \cong 6.07$ minutes.

17.

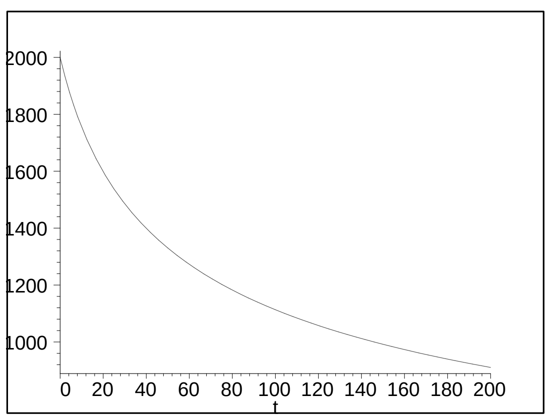
(a) The solution of this separable equation is given by

$$u^3 = \frac{u_0^3}{3\alpha u_0^3 t + 1}.$$

Since $u_0 = 2000$, the specific solution is

$$u(t) = \frac{2000}{(6t/125 + 1)^{1/3}}.$$

(b)

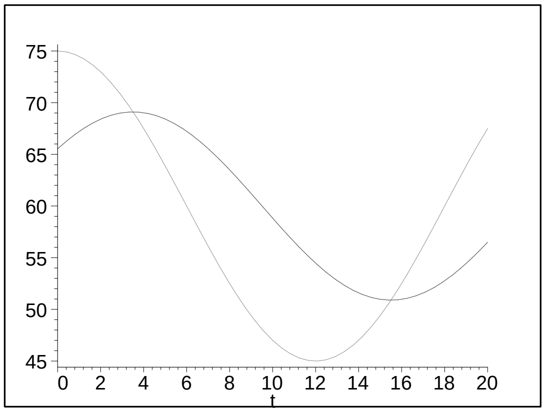


(c) We look for τ so that $u(\tau) = 600$. The solution of this equation is $t \approx 750.77$ seconds.

18.

(a) The integrating factor is $\mu(t) = e^{kt}$. Then $u = e^{-kt} \int ke^{kt}(T_0 + T_1 \cos(\omega t)) = ce^{-kt} + T_0 + kT_1(k \cos \omega t + \omega \sin \omega t)/(k^2 + \omega^2)$. Since $e^{-kt} \to 0$ as $t \to \infty$, we see that the steady state is $S(t) = T_0 + kT_1(k \cos(\omega t) + \omega \sin(\omega t))/(k^2 + \omega^2)$.

(b)



The amplitude R of the oscillatory part of S(t) is approximately 9 degrees Fahrenheit. The time lag τ between maxima is approximately 3.5 seconds.

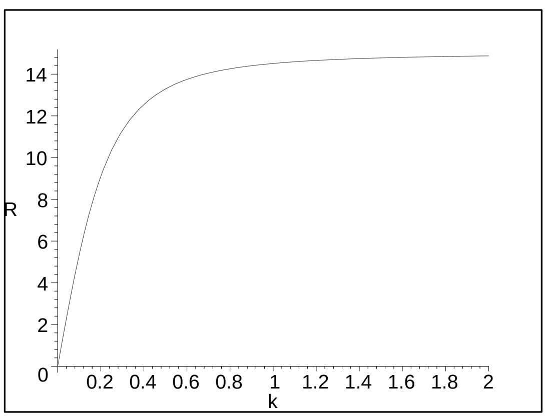
(c) From above, the oscillatory part of $S(t)$ is given by

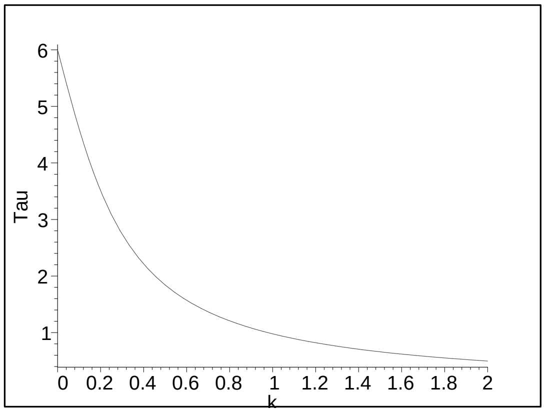
$$kT_1 \frac{k\cos(\omega t) + \omega \sin(\omega t)}{k^2 + \omega^2} = \frac{kT_1}{\sqrt{k^2+\omega^2}}(\cos(\omega t)\cos(\omega\tau) + \sin(\omega t)\sin(\omega\tau))$$

for τ such that $\cos(\omega\tau) = k/\sqrt{k^2+\omega^2}$ and $\sin(\omega\tau) = \omega/\sqrt{k^2+\omega^2}$. That is, $\tau = \frac{1}{\omega}\arctan(\omega/k)$. Further, letting $R = kT_1/\sqrt{k^2+\omega^2}$, we can write the oscillatory part of $S(t)$ as

$$R[\cos(\omega t)\cos(\omega\tau) + \sin(\omega t)\sin(\omega\tau)] = R\cos(\omega(t-\tau)).$$

Below we show graphs of R and τ versus k.





19.

(a) The differential equation for Q is

$$\frac{dQ}{dt} = kr + P - \frac{Q(t)}{V}r.$$

Therefore,

$$V\frac{dc}{dt} = kr + P - c(t)r.$$

The solution of this equation is $c(t) = k + P/r + (c_0 - k - P/r)e^{-rt/V}$. As $t \to \infty$, $c(t) \to k + P/r$.

(b) In this case, we will have $c(t) = c_0 e^{-rt/V}$. The reduction times are $T_{50} = \ln(2)V/r$ and $T_{10} = \ln(10)V/r$.

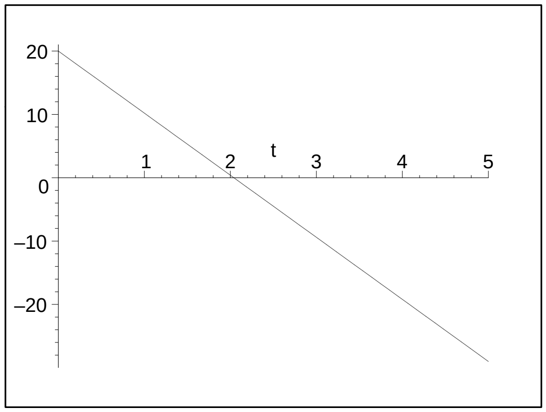
(c) Using the results from part (b), we have: Superior, $T = 431$ years; Michigan, $T = 71.4$ years; Erie, $T = 6.05$ years; Ontario, $T = 17.6$ years.

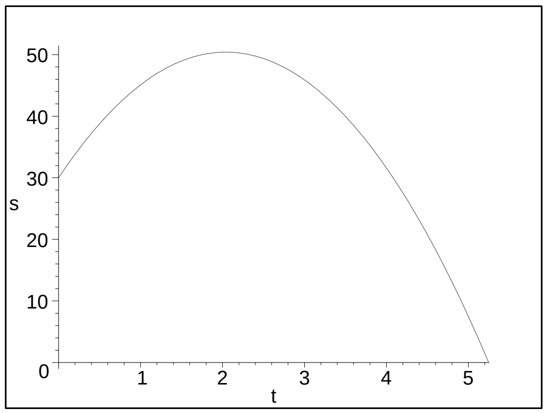
20.

(a) Assuming no air resistance, we have $dv/dt = -9.8$. Therefore, $v(t) = -9.8t + v_0 = -9.8t + 20$ and its position at time t is given by $s(t) = -4.9t^2 + 20t + 30$. When the ball reaches its max height, the velocity will be zero. We see that $v(t) = 0$ implies $t = 20/9.8 \cong 2.04$ seconds. When $t = 2.04$, we see that $s(2.04) \cong 50.4$ meters.

(b) Solving $s(t) = -4.9t^2 + 20t + 30 = 0$, we see that $t = 5.248$ seconds.

(c)



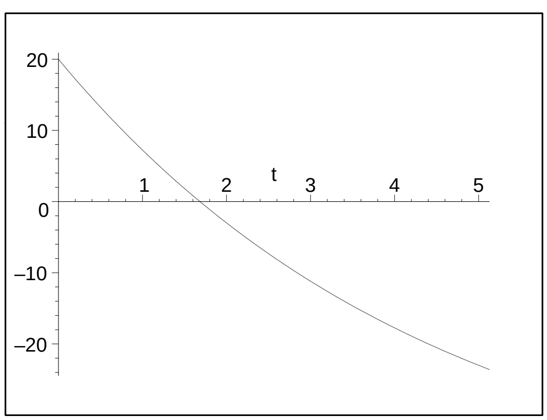


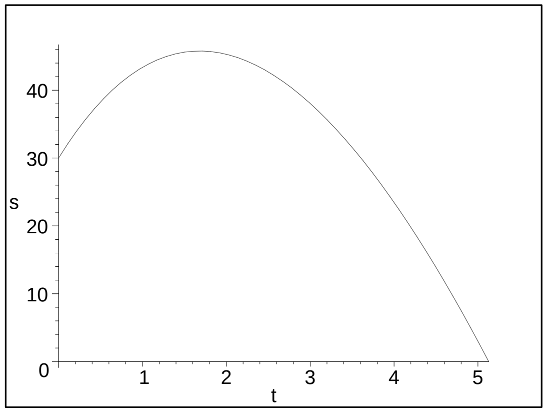
21.

(a) We have $mdv/dt = -v/30 - mg$. Given the conditions from problem 20, we see that the solution is given by $v(t) = -44.1 + 64.1e^{-t/4.5}$. The ball will reach its max height when $v(t) = 0$. This occurs at $t = 1.683$ seconds. The height of the ball is given by $s(t) = -318.45 - 44.1t - 288.45e^{-t/4.5}$. When $t = 1.683$, we have $s(1.683) = 45.78$ meters, the maximum height.

(b) The ball will hit the ground when $s(t) = 0$. This occurs when $t = 5.128$ seconds.

(c)



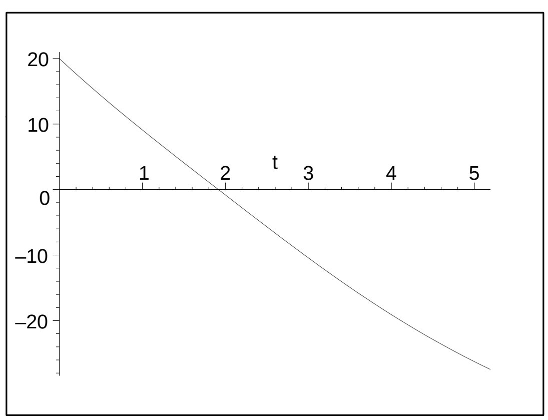


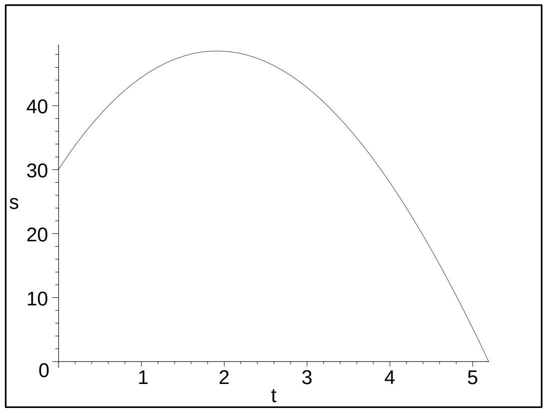
22.

(a) The equation for the upward motion is $mdv/dt = -\mu v^2 - mg$ where $\mu = 1/1325$. Using the data from exercise 20, and the fact that this equation is separable, we see its solution is given by $v(t) = 44.133\tan(.425 - .222t)$. Setting $v(t) = 0$, we see the ball will reach its max height at $t = 1.916$ seconds. Integrating $v(t)$, we see the position at time t is given by $s(t) = 198.75\ln(\cos(0.222t - 0.425)) + 48.57$. Therefore, the max height is given by $s(1.916) = 48.56$ meters.

(b) The differential equation for the downward motion is $mdv/dt = +\mu v^2 - mg$. The solution of this equation is given by $v(T) = 44.13(1 - e^{t/2.25})/(1 + e^{t/2.25})$. Integrating $v(t)$, we see that the position is given by $s(t) = 99.29\ln(e^{t/2.25}/(1 + e^{t/2.25})^2) + 186.2$. Setting $s(t) = 0$, we see that the ball will spend $t = 3.276$ seconds going downward before hitting the ground. Combining this time with the amount of time the ball spends going upward, 1.916 seconds, we conclude that the ball will hit the ground 5.192 seconds after being thrown upward.

(c)





23.

(a) Measure the positive direction of motion downward. Then the equation of motion is given by

$$m\frac{dv}{dt} = \begin{cases} -0.75v + mg & 0 < t < 10 \\ -12v + mg & t > 10. \end{cases}$$

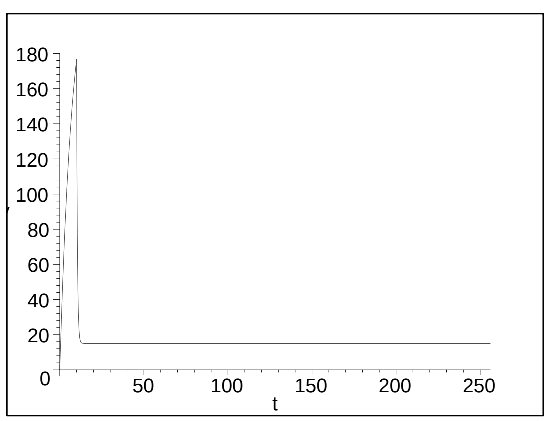
For the first 10 seconds, the equation becomes $dv/dt = -v/7.5 + 32$ which has solution $v(t) = 240(1 - e^{-t/7.5})$. Therefore, $v(10) = 176.7$ feet per second.

(b) Integrating the velocity function from part (a), we see that the height of the skydiver at time t $(0 < t < 10)$ is given by $s(t) = 240t + 1800e^{-t/7.5} - 1800$. Therefore, $s(10) = 1074.5$ feet.

(c) After the parachute opens, the equation for v is given by $dv/dt = -32v/15 + 32$ (as discussed in part (a)). We will reset t to zero. The solution of this differential equation is given by $v(t) = 15 + 161.7e^{-32t/15}$. As $t \to \infty$, $v(t) \to 15$. Therefore, the limiting velocity is $v_l = 15$ feet/second.

(d) Integrating the velocity function from part (c), we see that the height of the sky diver after falling t seconds with his parachute open is given by $s(t) = 15t - 75.8e^{-32t/15} + 1150.3$. To find how long the skydiver is in the air after the parachute opens, we find T such that $s(T) = 0$. Solving this equation, we have $T = 256.6$ seconds.

(e)



24.

(a) The equation of motion is given by $dv/dx = -\mu v$.

(b) The speed of the sled satisfies $\ln(v/v_0) = -\mu x$. Therefore, μ must satisfy $\ln(15/150) = -2000\mu$. Therefore, $\mu = \ln(10)/2000$ f^{-1}.

(c) The solution of $dv/dt = -\mu v^2$ can be expressed as $1/v - 1/v_0 = \mu t$. Using the fact that 1 mi/hour $\approx$ 1.467 feet/second, the elapsed time is $t \approx 35.56$ seconds.

25.

(a) Measure the positive direction of motion upward. The equation of motion is given by $mdv/dt = -kv - mg$. The solution of this equation is given by $v(t) = -mg/k + (v_0 + mg/k)e^{-kt/m}$. Solving $v(t) = 0$, we see that the mass will reach its max height $t_m = (m/k)\ln[(mg + kv_0)/mg]$ seconds after being projected upward. Integrating the velocity equation, we see that the position of the mass at this time will be given by the position equation

$$s(t) = -mgt/k + \left[\left(\frac{m}{k}\right)^2 g + \frac{mv_0}{k}\right](1 - e^{-kt/m}).$$

Therefore, the max height reached is

$$x_m = s(t_m) = \frac{mv_0}{k} - g\left(\frac{m}{k}\right)^2 \ln\left[\frac{mg + kv_0}{mg}\right].$$

,

(b) These formulas for t_m and x_m come from the fact that for $\delta << 1$, $\ln(1+\delta) = \delta - \frac{1}{2}\delta^2 + \frac{1}{3}\delta^3 - \frac{1}{4}\delta^4 + \ldots$. This formula is just Taylor's formula.

(c) Consider the result for t_m in part (b). Multiplying the equation by $\frac{g}{v_0}$, we have

$$\frac{t_m g}{v_0} = \left[1 - \frac{1}{2}\frac{kv_0}{mg} + \frac{1}{3}\left(\frac{kv_0}{mg}\right)^2 - \ldots\right].$$

The units on the left, must match the units on the right. Since the units for $t_m g/v_0 = (s \cdot m/s^2)/(m/s)$, the units cancel. As a result, we can conclude that kv_0/mg is dimensionless.

26.

(a) The equation of motion is given by $mdv/dt = -kv - mg$. The solution of this equation is given by $v(t) = -mg/k + (v_0 + mg/k)e^{-kt/m}$.

(b) Applying L'Hospital's rule, as $k \to 0$, we have

$$\lim_{k\to 0} -mg/k + (v_0 + mg/k)e^{-kt/m} = v_0 - gt.$$

(c)

$$\lim_{m\to 0} -mg/k + (v_0 + mg/k)e^{-kt/m} = 0.$$

27.

(a) The equation of motion is given by

$$m\frac{dv}{dt} = -6\pi\mu av + \rho'\frac{4}{3}\pi a^3 g - \rho\frac{4}{3}\pi a^3 g.$$

We can rewrite this equation as

$$v' + \frac{6\pi\mu a}{m}v = \frac{4}{3}\frac{\pi a^3 g}{m}(\rho' - \rho).$$

Multiplying by the integrating factor $e^{6\pi\mu at/m}$, we have

$$(e^{6\pi\mu at/m}v)' = \frac{4}{3}\frac{\pi a^3 g}{m}(\rho' - \rho)e^{6\pi\mu at/m}.$$

Integrating this equation, we have

$$\begin{aligned} v &= e^{-6\pi\mu at/m}\left[\frac{2a^2 g(\rho' - \rho)}{9\mu}e^{6\pi\mu at/m} + C\right] \\ &= \frac{2a^2 g(\rho' - \rho)}{9\mu} + Ce^{-6\pi\mu at/m}. \end{aligned}$$

Therefore, we conclude that the limiting velocity is $v_L = (2a^2 g(\rho' - \rho))/9\mu$.

(b) By the equation above, we see that the force exerted on the droplet of oil is given by

$$Ee = -6\pi\mu av + \rho'\frac{4}{3}\pi a^3 g - \rho\frac{4}{3}\pi a^3 g.$$

If $v = 0$, then solving the above equation for e, we have

$$e = \frac{4\pi a^3 g(\rho' - \rho)}{3E}.$$

28.

(a) The equation is given by $mdv/dt = -kv - mg$. The solution of this equation is $v(t) = -(mg/k)(1 - e^{-kt/m})$. Integrating, we see that the position function is given by $x(t) = -(mg/k)t + (m/k)^2 g(1 - e^{-kt/m}) + 30$. First, by setting $x(t) = 0$, we see that the ball will hit the ground $t = 3.63$ seconds after it is dropped. Then $v(3.63) = 11.58$ m/second will be the speed when the mass hits the ground.

(b) In terms of displacement, we have $mvdv/dx = -kv + mg$. This equation comes from applying the chain rule: $dv/dt = dv/dx \cdot dx/dt = vdv/dx$. The solution of this differential equation is given by

$$x(v) = -\frac{mv}{k} - \frac{m^2 g}{k^2} \ln \left| \frac{mg - kv}{mg} \right|.$$

Plugging in the given values for k, m, g, we have $x(v) = -1.25v - 15.31 \ln |0.0816v - 1|$. If $v = 10$, then $x(10) = 13.45$ meters.

(c) Using the equation for $x(v)$ above, we set $x(v) = 30$, $v = 10$, $m = 0.25$, $g = 9.8$. Then solving for k, we have $k = 0.24$.

29.

(a) The equation of motion is given by $mdv/dt = -GMm/(R+x)^2$. By the chain rule,

$$m\frac{dv}{dx} \cdot \frac{dx}{dt} = -G\frac{Mm}{(R+x)^2}.$$

Therefore,

$$mv\frac{dv}{dx} = -G\frac{Mm}{(R+x)^2}.$$

This equation is separable with solution $v^2 = 2GM(R+x)^{-1} + 2gR - 2GM/R$. Here we have used the initial condition $v_0 = \sqrt{2gR}$. From physics, we know that $g = GM/R^2$. Using this substitution, we conclude that $v(x) = \sqrt{2g}[R/\sqrt{R+x}]$.

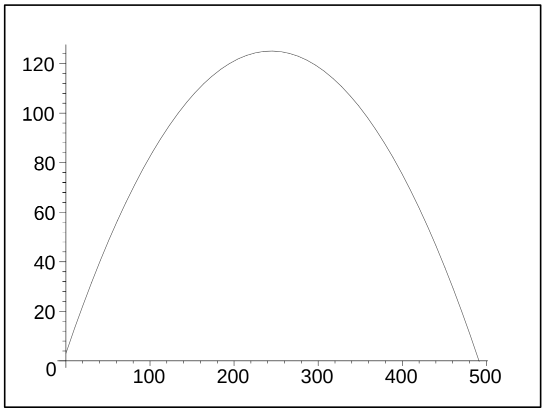
(b) By part (a), we know that $dx/dt = v(x) = \sqrt{2g}[R/\sqrt{R+x}]$. We want to solve this differential equation with the initial condition $x(0) = 0$. This equation is separable with solution $x(t) = [\frac{3}{2}(\sqrt{2g}Rt + \frac{2}{3}R^{3/2}]^{2/3} - R$. We want to find the time T such that $x(T) = 240,000$. Solving this equation, we conclude that $T \approx 50.6$ hours.

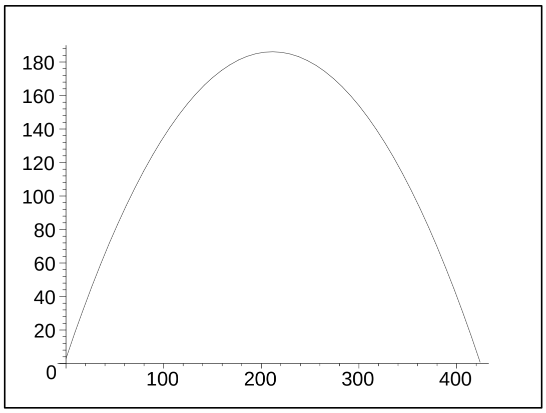
30.

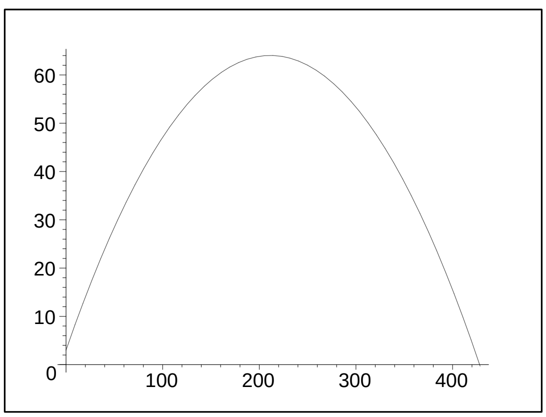
(a) $dv/dt = 0$ implies v is constant, but clearly by the initial condition $v = u \cos A$. $dw/dt = -g$ implies $w = -gt + C$, but also by the initial condition $w = -gt + u \sin A$.

(b) The equation $dx/dt = v = u \cos A$ along with the initial condition implies $x(t) = u \cos At$. The equation $dy/dt = w = -gt + u \sin A$ along with the initial condition implies $y(t) = -gt^2/2 + u \sin At + h$.

(c) Below we have plotted the trajectory of the ball in the cases $\pi/4$, $\pi/3$ and $\pi/6$ respectively.







(d) First, let T be the time it takes for the ball to travel L feet horizontally. Using the equation for x, we know that $x(T) = u \cos AT = L$ implies $T = L/u \cos A$. Then, when the ball reaches this wall, we need the height of the ball to be at least H feet. That is, we need $y(T) \geq H$. Now $y(t) = -16t^2 + u \sin At + 3$ implies we need $y(T) = -16L^2/(u^2 \cos^2(A)) + L \tan A + 3 \geq H$.

(e) If $L = 350$ and $H = 10$, then our inequality becomes

$$-\frac{1,960,000}{(u^2 \cos^2(A))} + 350 \tan A + 3 \geq 10.$$

Now if $u = 110$, then our inequality becomes

$$-\frac{162}{\cos^2(A)} + 350\tan(A) \geq 7.$$

Solving this inequality, we conclude that $0.63\ \text{rad} \leq A \leq 0.96\ \text{rad}$.

(f) We rewrite the inequality in part (c) as

$$\cos^2(A)(350\tan A - 7) \geq \frac{1,960,000}{u^2}.$$

In order to determine the minimum value necessary, we will maximize the function on the left-hand side. Letting $f(A) = \cos^2(A)(350\tan A - 7)$, we see that $f'(A) = 350\cos(2A) + 7\sin(2A)$. Therefore, $f'(A) = 0$ implies $\tan(2A) = -50$. For $0 < A < \pi/2$, we see that this occurs at $A = 0.7954$ radians. Substituting this value for A into the inequality above, we conclude that

$$u^2 \geq 11426.24.$$

Therefore, the minimum velocity necessary is 106.89 mph and the optimal angle necessary is 0.7954 radians.

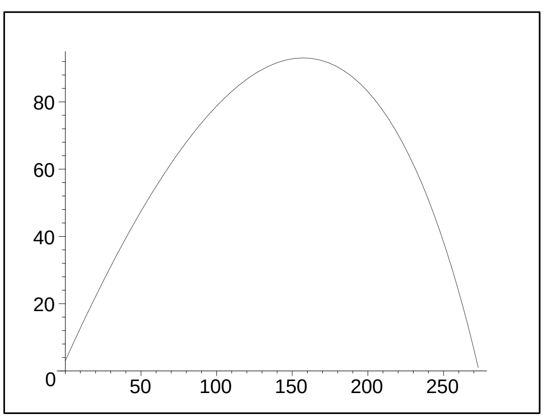
31.

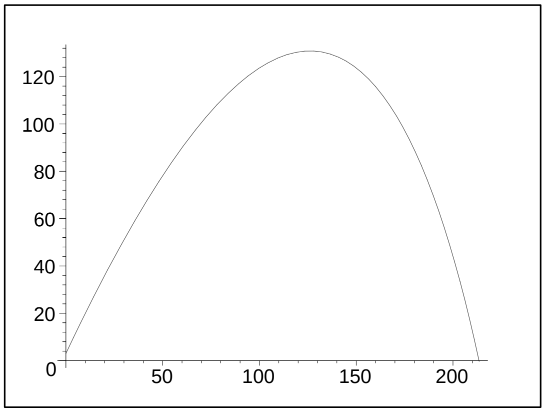
(a) The initial conditions are $v(0) = u\cos(A)$ and $w(0) = u\sin(A)$. Therefore, the solutions of the two equations are $v(t) = u\cos(A)e^{-rt}$ and $w(t) = -g/r + (u\sin(A) + g/r)e^{-rt}$.

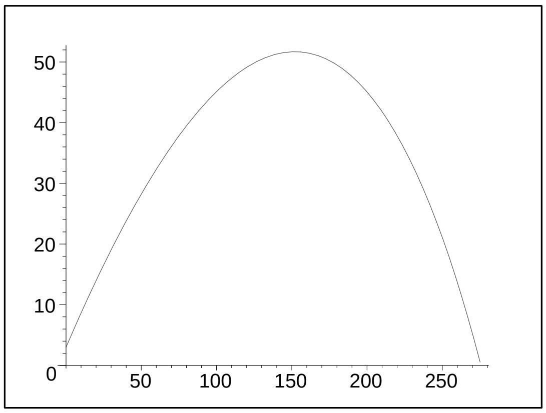
(b) Now $x(t) = \int v(t) = \frac{u}{r}\cos(A)(1 - e^{-rt})$, and

$$y(t) = \int w(t) = -\frac{gt}{r} + \frac{(g + ur\sin(A) + hr^2)}{r^2} - \left(\frac{u}{r}\sin(A) + \frac{g}{r^2}\right)e^{-rt}.$$

(c) Below we show trajectories for the cases $A = \pi/4, \pi/3$ and $\pi/6$, respectively.







(d) Let T be the time it takes the ball to go 350 feet horizontally. Then from above, we see that $e^{-T/5} = (u\cos(A) - 70)/u\cos(A)$. At the same time, the height of the ball is given by $y(T) = -160T + 267 + 125u\sin(A) - (800 + 5u\sin(A))[(u\cos(A) - 70)/u\cos(A)]$. Therefore, u and A must satisfy the inequality

$$800\ln\left[\frac{u\cos(A) - 70}{u\cos(A)}\right] + 267 + 125u\sin(A) - (800 + 5u\sin(A))\left[\frac{u\cos(A) - 70}{u\cos(A)}\right] \geq 10.$$

32.

(a) Solving equation (i), we have $y'(x) = [(k^2 - y)/y]^{1/2}$. The positive answer is chosen since y is an increasing function of x.

(b) $y = k^2\sin^2 t \implies dy/dt = 2k^2\sin t\cos t$. Substituting this into the equation in part (a), we have

$$\frac{2k^2\sin t\cos t dt}{dx} = \frac{\cos t}{\sin t}.$$

Therefore, $2k^2\sin^2 t dt = dx$.

(c) Letting $\theta = 2t$, we have $k^2\sin^2(\theta/2)d\theta = dx$. Integrating both sides, we have $x(\theta) = k^2(\theta - \sin\theta)/2$. Further, using the fact that $y = k^2\sin^2 t$, we conclude that $y = k^2\sin^2(\theta/2) = k^2(1 - \cos(\theta))/2$.

(d) From part (c), we see that $y/x = (1 - \cos\theta)/(\theta - \sin\theta)$. If $x = 1$ and $y = 2$, the solution of the equation is $\theta \approx 1.401$. Substituting that value of θ into either of the equations in part (c), we conclude that $k \approx 2.193$.

[redacted]